RECONSTRUCTION OF THE SHAPE OF A SOLID OF REVOLUTION FROM A GIVEN VELOCITY DISTRIBUTION OF A FLOW ALONG ITS SURFACE

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We consider the problem of finding the shape of a solid of revolution, given the magnitude of a flow velocity on it as a function of arc length along the generatrix. The flow is assumed to be rotationless, steady-state, and axially symmetric, and the fluid is assumed to be ideal and incompressible. The problem is solved in an exact, nonlinear formulation. In contrast with the work of Kiselev [1] and Éterman [2] the proposed method enables us to get a solution to any given degree of accuracy.

In an axially symmetric, rotationless flow the Stokes stream function $\Psi(x, r)$ satisfies the equation

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r}=0,$$

where x and r are cylindrical coordinates.

The stream function is defined up to an arbitrary constant. The constant is chosen from the condition that Ψ go to zero along the unknown boundary of the solid, the equation of this boundary being $r = \rho(s)$, $0 \leq s \leq L$, where s is the arc length along the generatrix.

The given form of the magnitude of the flow velocity at the solid as a function of the arc length,

$$v=v_\infty V(s),\ 0\leqslant s\leqslant L \tag{1}$$

serves as an additional condition determining $\rho(s)$. Here v_∞ is the magnitude of the unperturbed flow velocity.

It should be noted that the problem is not solvable for any dependence V(s). A necessary condition that the problem be solvable is that $\max v(s) > v_\infty$. For a proof of this fact we consider in addition to the basic flow a uniform flow with velocity $v = v_\infty$ outside a cylindrical tube of infinite length and diameter equal to the maximum dimension of the solid. By construction, the domain of the basic flow includes the domain of the auxiliary flow. Hence, for these flows all the conditions of the Lavrent'ev comparison theorem [3] are satisfied, from which it follows that at the points of contact of the boundary streamlines the velocity of the basic flow is greater than the velocity of the auxiliary flow. Thus, at the boundary of the initial flow there is a point with arc length $s = s_*$ for which $v(s_*) > v_\infty$.

We represent the initial flow as the result of imposing a uniform flow with velocity v_∞ on a system of vortex rings of line intensity $\gamma(s)$, continuously distributed along the boundary of the obstructing solid. The stream function in this case is of the form [4]

$$\Psi(x,r)=\frac{v_\infty r^2}{2}-\frac{r}{4\pi}\iint\limits_{\Omega}\frac{\gamma(s)\cos(\theta-\varphi)\,d\omega}{R}, \tag{2}$$

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where $d\omega = \rho(s)dsd\varphi$ is the surface element of the surface Ω; $R = \sqrt{(x-\xi)^2 + r^2 + \rho^2 - 2r\rho\cos(\theta - \varphi)}$. Because of the axial symmetry we set $\theta = 0$. Using the fact that [4]

$$\int_0^{2\pi} \frac{\cos\varphi d\varphi}{R} = \frac{\sqrt{(x-\xi)^2 + (r+\rho)^2}}{r\rho}[(2-\lambda^2)K(\lambda) - 2E(\lambda)],$$

where $K(\lambda)$ and $E(\lambda)$ are the complete elliptical integrals of the first and second kinds with modulus $\lambda = 2\sqrt{r\rho}[x-\xi)^2 + (r+\rho)^2]^{-1}$, we obtain from (2) the following representation of the stream function:

$$\Psi(x, r) = \frac{v_\infty r^2}{2} - \frac{1}{4\pi}\int_0^L \gamma(s)\sqrt{(x-\xi)^2 + (r+\rho)^2}[(2-\lambda^2)K(\lambda) - 2E(\lambda)]\,ds.$$

Keeping in mind that the line intensity $\gamma(s)$ of the vortex layer is equal to the magnitude of the flow velocity at the surface of the solid [5], from the condition $\Psi(\xi, \rho) = 0$ at the boundary we obtain the following integrodifferential equation for $\rho(s)$:

$$\rho(s) = \frac{1}{2\pi\rho(s)}\int_0^1 V(\sigma)\sqrt{[\xi(\sigma) - \xi(s)]^2 + [\rho(\sigma) + \rho(s)]^2}\,[(2-\lambda^2)K(\lambda) - 2E(\lambda)]\,d\sigma \equiv A[\rho], \quad (3)$$

where

$$\xi(s) = \int_0^s \sqrt{1 - (d\rho/d\sigma)^2}\,d\sigma, \quad (4)$$

$$\lambda = \lambda(\sigma, s) = 2\sqrt{\rho(\sigma)\rho(s)}/\tau(\sigma, s),$$

$$\tau(\sigma, s) = \sqrt{[\xi(\sigma) - \xi(s)]^2 + [\rho(\sigma) + \rho(s)]^2}.$$

We have gone to dimensionless variables in this equation. As a characteristic linear dimenion and a characteristic velocity we take the overall length L of the generatrix of the solid and the velocity v_∞ of the unperturbed flow, respectively.

We represent the unknown function $\rho(s)$ as a third-order spline. The coefficients of the cubic polynomial at each interval $[s_k, s_{k+1}]$ $(k = 0, 1, \ldots, n-1)$ are determined from the continuity of the function and its first two derivatives at the nodal points [6, 7].

The integrodifferential equation (3) is solved by the method of successive approximation. As the zeroth approximation we can take $\rho^{(0)}(s) = s(1-s)$. The value $\rho_k^1 = \rho^{(1)}(s_k)$ of the unknown function at the nodal pints s_k at the first step of the iteration process is found from Eq. (3)

$$\rho_k^1 = A\left[\rho^{(0)}\right]_{s=s_k}.$$

After the parameters ρ_k^1 $(k = 1, 2, \ldots, n-1)$ are found, the problem is solved by interpolating the function $\rho^{(1)}(s)$ with the aid of the cubic spline. The function $\xi = \xi^{(1)}(s)$ is also represented as a third-order spline. The quantities $\xi_k^1 = \xi^{(1)}(s_k)$ are found from Eq. (4).

The subsequent approximations are constructed in a completely analogous manner, and the iteration process is continued until the condition

$$\sum_{k=1}^{n-1} |\rho_k^{i+1} - \rho_k^i| < \varepsilon$$

is met.

The accuracy is monitored by solving the direct problem of flow around the obtained solid of revolution and then comparing the so-obtained velocity distribution along its boundary with the initial dependence (1). The required accuracy of the calculations is achieved by increasing the number of nodal points.

In the numerical calculations the logarithmic singularity of the integrand in Eq. (3) at $\sigma = s$ is separated out,

TABLE 1

ρ_{ex}	ρ_{num}	ξ_{ex}	ξ_{num}
0,09836	0,09841	0,01558	0,01567
0,18710	0,18719	0,06079	0,06070
0,25752	0,25756	0,13121	0,13119
0,30273	0,30274	0,21995	0,21994
0,31831	0,31830	0,31831	0,31831

$$\rho(s) = \frac{1}{2\pi\rho(s)}\left\{\int_0^1 V(\sigma)\tau(\sigma, s)[(2-\lambda^2)F(\lambda) - 2E(\lambda)]\,d\sigma - \right.$$

$$\left. - \int_0^1 V(\sigma)\tau(\sigma, s)(2-\lambda^2)\ln|\sigma - s|\,d\sigma\right\},$$

where $F(\lambda) = K(\lambda) + \ln|\sigma - s|$ has no singularities on the interval [0,1].

The improper integrals are computed by Gaussian quadrature. When integrating a function with logarithmic singularity the Gaussian quadrature formula is used with weight $g(u) = \ln u$

$$\int_0^1 f(u)\ln u\,du = -\sum_{k=1}^{10} A_k f(u_k).$$

A table of the nodes u_k and coefficients A_k is given by Krylov and Pal'tsev [8].

For the complete elliptical integrals we use the approximate equations

$$K(\lambda) = \ln 4 - \frac{1}{2}\ln p(\sigma, s) - \ln|\sigma - s| + \sum_{k=1}^{4}\eta^k(a_k - b_k\ln\eta);$$

$$E(\lambda) = 1 + \sum_{k=1}^{4}\eta^k(c_k - d_k\ln\eta),$$

where

$$\eta = 1 - \lambda^2;\quad p(\sigma, s) = \frac{1}{\tau^2(\sigma, s)}\left\{\left[\frac{\xi(\sigma) - \xi(s)}{\sigma - s}\right]^2 + \left[\frac{\rho(\sigma) - \rho(s)}{\sigma - s}\right]^2\right\},$$

and the numerical values of the parameters a_k, b_k, c_k, and d_k ($k = 1, 2, 3, 4$) are given by Dymarskii et al. [9]. The maximum error in these representations is less than $1.5 \cdot 10^{-9}$.

As an example let us consider a velocity of the form

$$V(s) = \frac{3}{2}\sin s\pi.$$

This velocity distribution is realized in the flow about a sphere [4]. Thus, the exact solution of the problem is of the form

$$\rho(s) = \frac{1}{\pi}\sin s\pi,$$

$$\xi(s) = \frac{1}{\pi}(1 - \cos s\pi).$$

An approximate solution was obtained for $n = 10$ and $\varepsilon = 0.0001$. The computation time on a BÉSM-6 computer was less than 1 min. In Table 1 we compare the approximate and exact solutions at equidistant points s_k ($k = 1, 2, 3, 4, 5$). Thus, the maximum deviation of the approximate solution from the exact solution is less than 0.0001 of the length of the generatrix L.

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